

# Dielectric black holes induced by a refractive index perturbation and the Hawking effect

**Citation for published version:**

Belgiorno, F, Cacciatori, SL, Ortenzi, G, Rizzi, L, Gorini, V & Faccio, D 2011, 'Dielectric black holes induced by a refractive index perturbation and the Hawking effect', *Physical Review D - Particles, Fields, Gravitation and Cosmology*, vol. 83, no. 2, 024015. <https://doi.org/10.1103/PhysRevD.83.024015>

**Digital Object Identifier (DOI):**

[10.1103/PhysRevD.83.024015](https://doi.org/10.1103/PhysRevD.83.024015)

**Link:**

[Link to publication record in Heriot-Watt Research Portal](#)

**Document Version:**

Publisher's PDF, also known as Version of record

**Published In:**

Physical Review D - Particles, Fields, Gravitation and Cosmology

**General rights**

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [open.access@hw.ac.uk](mailto:open.access@hw.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.

**Dielectric black holes induced by a refractive index perturbation and the Hawking effect**F. Belgiorno,<sup>1</sup> S. L. Cacciatori,<sup>2,3</sup> G. Ortenzi,<sup>4</sup> L. Rizzi,<sup>2</sup> V. Gorini,<sup>2,3</sup> and D. Faccio<sup>5</sup><sup>1</sup>*Dipartimento di Fisica, Università di Milano, Via Celoria 16, IT-20133 Milano, Italy*<sup>2</sup>*Department of Physics and Mathematics, Università dell'Insubria, Via Valleggio 11, IT-22100 Como, Italy*<sup>3</sup>*INFN sezione di Milano, via Celoria 16, IT-20133 Milano, Italy*<sup>4</sup>*Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Cozzi 53, IT-20125 Milano, Italy*<sup>5</sup>*School of Engineering and Physical Sciences, SUPA, Heriot-Watt University, Edinburgh, Scotland EH14 4AS, UK*

(Received 22 March 2010; revised manuscript received 18 November 2010; published 13 January 2011)

We consider a 4D model for photon production induced by a refractive index perturbation in a dielectric medium. We show that, in this model, we can infer the presence of a Hawking type effect. This prediction shows up both in the analogue Hawking framework, which is implemented in the pulse frame and relies on the peculiar properties of the effective geometry in which quantum fields propagate, as well as in the laboratory frame, through standard quantum field theory calculations. Effects of optical dispersion are also taken into account, and are shown to provide a limited energy bandwidth for the emission of Hawking radiation.

DOI: [10.1103/PhysRevD.83.024015](https://doi.org/10.1103/PhysRevD.83.024015)

PACS numbers: 04.62.+v, 04.70.Dy

**I. INTRODUCTION**

One of the most intriguing predictions of quantum fields in curved spacetime geometries is the production of Hawking radiation. In 1974 S. Hawking predicted that black holes emit particles with a thermal spectrum. Therefore a black hole will evaporate, shedding energy under the form of a blackbody emission [1–3]. However, it turns out that for a typical stellar mass black hole the temperature of this radiation is so low ( $\sim 10$  nK) that it has no hope of being directly detected. Notwithstanding, Hawking radiation does not actually require a true gravitational event horizon but, rather, it is essentially a kinematical effect, i.e. it requires some basic “kinematical” ingredients but no specific underlying dynamics (see e.g. [4–6]). What suffices is the formation of a trapping horizon in a curved spacetime metric of any kind, and the analysis of the behavior of any quantum field therein (see also the discussion in [6]). The quanta of this field will then be excited according to the prediction of Hawking. On this basis, a number of analogue systems have been proposed, for the first time by W. Unruh [7] and later by other researchers (see e.g. [8] and references therein), that aim at reproducing the kinematics of gravitational systems. Most of these analogies rely on acoustic perturbations and on the realization of so-called dumb holes: a liquid or gas medium is made to flow faster than the velocity of acoustic waves in the same medium so that at the transition point between sub and supersonic flow, a trapping horizon is formed that may be traversed by the acoustic quanta, viz. phonons, only in one direction [8]. Unfortunately the phonon blackbody spectrum is expected to still have very low temperatures, thus eluding direct detection (see e.g. [9]). A parallel line of investigation involves effective geometry for light, which has been introduced by Gordon [10] and extended also to nonlinear electrodynamics; black hole metrics have been introduced and the possibility to perform

experiments involving Hawking analogue radiation has been explored [11–23]. Recently Philbin *et al.* proposed an optical analogue in which a soliton with intensity  $I$ , propagating in an optical fibre, generates through the nonlinear Kerr effect a refractive index perturbation (RIP),  $\delta n = n_2 I$ , where  $n_2$  is the Kerr index [24]. The same mechanism has also been generalized to a full 4D geometry by Faccio *et al.* [25] and has led to the first experimental observation of quanta emitted from an analogue horizon [26]. The RIP modifies the spacetime geometry as seen by copropagating light rays and, similarly to the acoustic analogy, if the RIP is locally superluminal, i.e. if it locally travels faster than the phase velocity of light in the medium, an horizon is formed and Hawking radiation is to be expected.

Here we take into consideration the Hawking effect in dielectric black holes induced by a RIP which propagates with constant velocity  $v$ . We first show that, under suitable conditions, we can remap the original Maxwell equations for nonlinear optics into a geometrical description, in analogy to what is done in the case of acoustic perturbations in condensed matter. Then we provide a model in which the presence of the Hawking radiation can be deduced even without recourse to the analogous model characterized by a curved spacetime geometry. Hawking radiation is a new phenomenon for nonlinear optics, never foreseen before, which could legitimately meet a sceptical attitude by the nonlinear optics community, because of a missing description of the phenomenon by means of the standard tools of quantum electrodynamics. As a consequence, it is important to provide also a corroboration of the analogue picture simply by using “standard” tools of quantum field theory (without any reference to the geometrical picture).

We then take into consideration the effects of optical dispersion, which give rise to relevant physical consequences on the quantum phenomenon of particle creation.

We first introduce a 2D reduction of the model in the presence of dispersion, and see how the dispersion relation is affected by the frequency-dependence of the refractive index. Then we discuss both phase velocity horizons and group velocity horizons, which can both play a relevant role in the physical situation at hand; their qualitative difference and the possibility to discriminate between them in experiments (numerical and/or laboratory-based) is considered.

The structure of the paper is as follows. In Sec. II we introduce our idealized model of a RIP propagating, in a nonlinear Kerr medium, with a constant velocity  $v$  in the  $x$  direction and infinitely extended in the transverse  $y$  and  $z$  directions. In the eikonal approximation, the model is embodied in a suitable wave equation for the generic component of the electric field propagating in the nonlinear medium affected by the RIP. We describe this propagation as taking place in an analogue spacetime metric, written in the pulse frame, where the metric is static and displays two horizons  $x_+$  and  $x_-$  for propagating photons, analogues to a black hole and a white hole horizons, respectively. By assigning to the black hole horizon a surface gravity in a standard way, we calculate the evaporation temperature  $T_+$  of the RIP in the latter frame, which turns out to be proportional to the absolute value of the derivative of the RIP's profile, evaluated at  $x_+$ . We also show that  $T_+$  is conformally invariant, as expected for the consistency of the model.

In Sec. III, we tackle the model from a different perspective, namely, within the framework of quantum field theory. We construct a complete set of positive frequency solutions  $\Phi_{\mathbf{k}_l}$  of the wave equation in the lab frame, in terms of which we quantize the field in the standard way. Then, we compare the  $\Phi_{\mathbf{k}_l}$  with the corresponding plane wave solutions of the wave equation in the absence of the RIP. This enables us to calculate the Bogoliubov coefficients, in terms of which we express the expectation value of the number operator of the outgoing quanta in the *in* vacuum state. As expected, this average is a thermal-like distribution displaying the typical  $\cos\theta$  temperature profile in the lab frame, where  $\theta$  is the emission angle with respect to the direction of motion of the RIP. We then display the transformation law which relates the temperatures in the pulse and in the lab frame, respectively, by means of the usual Doppler formula.

The results worked out in the preceding sections apply in the approximation in which dispersion is neglected. In Sec. IV we study the modifications we expect when dispersion is taken into account. In particular, we show that, whereas in the dispersionless case one should expect to observe all the blackbody spectrum, in the presence of dispersion only a limited spectral region, which depends on  $v$ , will be excited, and the blackbody spectrum shape will not be discernible. Moreover, one can introduce two different concepts of horizon: phase velocity horizons and

group velocity horizons, which are shown to be involved with different spectral bandwidths, and with very different qualitative behavior regarding their action on photons.

Section V is devoted to the conclusions.

Finally, we have added three Appendixes. The first establishes a correspondence between our black hole metric and the acoustic one. The second one relates the temperature to the standard conical singularity of the Euclidean version of the metric. The third establishes the relevant analytic properties of the Bogoliubov coefficients.

## II. STATIC DIELECTRIC BLACK HOLE

We consider a model-equation which is derived from nonlinear electrodynamics (with  $\chi_2 = 0$  and  $\chi_3 \neq 0$ ) in the eikonal approximation for a perturbation of a full-nonlinear electric field propagating in a nonlinear Kerr medium. In order to make the forthcoming analysis as simple as possible we replace the electric field with a scalar field  $\Phi$  (cf. e.g. Schwinger's analysis of sonoluminescence [27]), for which we obtain the wave equation [28]

$$\frac{n^2(x_l - vt_l)}{c^2} \partial_{t_l}^2 \Phi - \partial_{x_l}^2 \Phi - \partial_y^2 \Phi - \partial_z^2 \Phi = 0. \quad (1)$$

Coordinates in the lab frame are denoted by  $t_l, x_l, y, z$  (the labels of  $y, z$  are omitted because they will not be involved in the boost relating the lab frame with the pulse frame). Here,  $n(x_l - vt_l)$  is the refractive index, which accounts for the propagating RIP in the dielectric. The latter can be obtained by means of an intense laser pulse in a nonlinear dielectric medium (Kerr effect). In our model the RIP does not depend on the transverse coordinates, namely, it is infinitely extended in the transverse dimensions. This approximation is necessary in order to carry out the calculations below, which allow us to draw a tight analogy between the Hawking geometrical description and the quantum field theoretical treatment. Such an infinitely extended RIP is clearly an idealization, which cannot be obtained in an actual experiment. However, relatively flat, e.g. super-Gaussian-like pulses may be produced that would, at least locally, fall within the approximations adopted here.

Equation (1) arises also in the eikonal approximation for a scalar field in the metric

$$ds^2 = \frac{c^2}{n^2(x_l - vt_l)} dt_l^2 - dx_l^2 - dy^2 - dz^2. \quad (2)$$

To carry out the analysis in the context of the analogue metric, we pass from the laboratory frame to the pulse frame (refractive index perturbation frame) by means of a boost:  $t = \gamma(t_l - \frac{v}{c}x_l)$ ,  $x = \gamma(x_l - vt_l)$ , and we obtain

$$ds^2 = c^2 \gamma^2 \frac{1}{n^2} \left(1 + \frac{nv}{c}\right) \left(1 - \frac{nv}{c}\right) dt^2 + 2\gamma^2 \frac{v}{n^2} (1 - n^2) dt dx - \gamma^2 \left(1 + \frac{v}{nc}\right) \left(1 - \frac{v}{nc}\right) dx^2 - dy^2 - dz^2. \quad (3)$$

We assume

$$n(x) = n_0 + \delta n = n_0 + \eta \bar{I}(x), \quad (4)$$

where  $\eta$  is meant to be positive, with  $\eta \ll 1$  due to the actual smallness of the Kerr index (a negative  $\eta$  could be easily taken into account);  $\bar{I}$  denotes the normalized intensity of the pulse, is taken to be a  $C^\infty$  function, rapidly decaying at infinity and with a single maximum at  $x = 0$ , of height 1. A scheme of the RIP is shown in Fig. 1. The form of  $\bar{I}$  implies that both  $\partial_t =: \xi$  and  $\partial_\phi$  are Killing vectors for the given metric. Then the surface  $g_{00} = 0$  is lightlike and corresponds to an event horizon. It is also possible to verify that the Frobenius integrability conditions are trivially satisfied, so that there exists a coordinate transformation carrying the metric into a static form (see e.g. [29]). Even if these coordinates are singular, we carry out the relative transformation because, on one hand, it allows a more straightforward comparison with the well-known Schwarzschild case, and on the other hand, it allows a direct computation of the greybody coefficient (cf. Sec. III D). To implement this transformation consider the following coordinate change:

$$dt = d\tau - \alpha(x)dx, \quad (5)$$

where

$$\alpha(x) = \frac{g_{01}(x)}{g_{00}(x)}. \quad (6)$$

Then the metric takes the static form

$$ds^2 = \frac{c^2}{n^2(x)} g_{\tau\tau}(x) d\tau^2 - \frac{1}{g_{\tau\tau}(x)} dx^2 - dy^2 - dz^2, \quad (7)$$

where

$$g_{\tau\tau}(x) := \gamma^2 \left( 1 + n(x) \frac{v}{c} \right) \left( 1 - n(x) \frac{v}{c} \right). \quad (8)$$

There is a remarkable resemblance of the  $\tau, x$  part of the metric with a standard static spherically symmetric metric in general relativity in the so-called Schwarzschild gauge, aside from the important difference represented by the factor  $\frac{c^2}{n^2}$  replacing  $c^2$ . The horizons are determined by the condition  $g_{\tau\tau} = 0$ , i.e. by

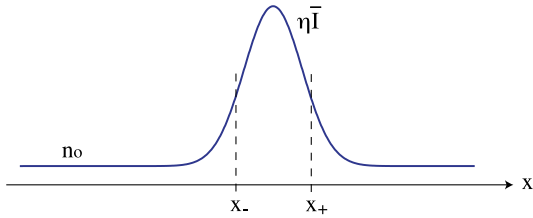


FIG. 1 (color online). Scheme of the RIP geometry.  $x_+$  and  $x_-$  indicate the black hole and white hole horizon positions, respectively.

$$1 - n(x) \frac{v}{c} = 0, \quad (9)$$

which, by Eq. (4), becomes  $n_0 + \eta \bar{I}(x) = c/v$ . Because of the specific shape assumed for  $\bar{I}$  there will be two horizons: one, denoted by  $x_+$ , located on the rising edge of the RIP (i.e.  $dn/dx|_{x_+} < 0$ ) and one, denoted by  $x_-$ , on the falling edge (i.e.  $dn/dx|_{x_-} > 0$ ). Since  $0 \leq \bar{I} \leq 1$ , the condition for the occurrence of the event horizons is

$$\frac{1}{n_0 + \eta} \leq \frac{v}{c} < \frac{1}{n_0}. \quad (10)$$

The external region corresponds to  $x < x_-$  and to  $x > x_+$ . The leading horizon  $x = x_+$  is a black hole horizon, whereas the trailing one  $x = x_-$  is a white hole horizon. This can be understood as follows. Consider the front observer, i.e. the observer toward which the pulse is moving. Any photon starting from the region behind the front zero  $x_+$  cannot reach this observer before the latter is reached by the pulse. Indeed, the pulse is moving superluminally relative to any photon travelling in the region between the two zeros  $x_-$  and  $x_+$ . Therefore, the “leading” zero  $x_+$  corresponds to a black hole horizon. Analogously, the “trailing” zero  $x_-$  corresponds to a white hole horizon. See also Appendix A for further details.

### A. Surface gravity and temperature

We can formally assign a temperature to the “black hole” horizon  $x_+$  by defining it in the usual way in terms of a “surface gravity”  $\kappa_+$  associated to the latter. We have

$$\begin{aligned} \kappa_+ &:= -c^4 \frac{1}{2} g^{ab} g^{cd} (\nabla_a \xi_c) (\nabla_b \xi_d) |_{x=x_+} \\ &= -\frac{c^4}{2} \left[ \frac{-2\gamma^4}{n^4} \left( \frac{dn}{dx} \right)^2 \right]_{x=x_+}, \end{aligned} \quad (11)$$

or [30]

$$\kappa_+ = \gamma^2 v^2 \left| \frac{dn}{dx}(x_+) \right|. \quad (12)$$

Then, the temperature is given by the familiar formula which has been also derived in [24]:

$$T_+ = \frac{\kappa_+ \hbar}{2\pi k_B c} = \gamma^2 v^2 \frac{\hbar}{2\pi k_B c} \left| \frac{dn}{dx}(x_+) \right|. \quad (13)$$

We can arrive at formula (12) by several different methods. For the sake of completeness, in Appendix B we employ a different method to derive the expression (13) for the temperature, and its invariance with respect to the conformal factor. The latter aspect is well-known in general relativity [31], and also in the frame of acoustic black holes [32]. Since

$$\frac{c}{v} = n(x_+) = n_0 + k\eta, \quad (14)$$

with  $k = \bar{I}(x_+) \in (0, 1)$ , we can rewrite Eqs. (12) and (13), respectively, as follows:



$$\kappa_+ = \frac{c^2}{(n_0 + k\eta)^2 - 1} \left| \frac{dn}{dx}(x_+) \right|, \quad (15)$$

$$T_+ = \frac{\hbar c}{2\pi k_b} \frac{1}{(n_0 + k\eta)^2 - 1} \left| \frac{dn}{dx}(x_0) \right|. \quad (16)$$

### B. Gaussian pulse

As an example of a refractive index perturbation  $\eta \bar{I}(x)$  consider a Gaussian normalized intensity of the pulse,

$$\bar{I}(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad (17)$$

and  $\eta$  a small parameter. These choice reflects a typical situation in experimental optics. Then the horizons given by Eq. (9)) occur for

$$x_{\pm} = \pm \sigma \sqrt{-2 \log \left[ \left( \frac{c}{v} - n_0 \right) \frac{1}{\eta} \right]}, \quad (18)$$

and  $x_- = -x_+$ . The surface gravity and the temperature are, respectively, given by

$$\kappa_+ = \gamma^2 v^2 \frac{k\eta}{\sigma} \sqrt{2 \log \frac{1}{k}}, \quad (19)$$

and by

$$T_+ = \frac{\kappa_+ \hbar}{2\pi k_b c} \sim \frac{\hbar c}{2\pi k_b} \frac{1}{n_0^2 - 1} \frac{k\eta}{\sigma} \sqrt{2 \log \frac{1}{k}}. \quad (20)$$

Typical values of the parameters are:  $\sigma \sim 10^{-5} \text{m}$ ,  $\eta \sim 10^{-3}$ ,  $n_0 \sim 1.45$ . Then, for  $k = \frac{1}{2}$ , say, we get

$$T_+ \sim 2 \times 10^{-2} \text{K}, \quad (21)$$

which is greater than in standard cases of sonic black holes. This temperature value is not to be meant as characteristic, somewhat larger values may be obtained. See the following subsection.

Note that this is the temperature in the frame of the pulse, with which the geometry is associated. A further transformation is needed in order to recover the temperature in the laboratory frame (see the following section). Note also that the temperature is proportional to the derivative with respect to  $x$  of the refractive index, evaluated at the black hole horizon. This is in agreement with the results of Ref. [24].

### C. Shockwave model

Let us assume that the rear part (trailing edge) of the filament is characterized by a shockwave profile (see e.g. [24]). This is a typical e.g. of spontaneous laser pulse dynamics in transparent media with third order nonlinearity. An analogous behavior is expected for the refractive index. As a consequence, in our model it is necessary to introduce, beyond the scale  $2\sigma$ , which, roughly speaking,

represents the overall spatial extension of the RIP, at least a further scale that corresponds to the “thickness” of the region where the refractive index undergoes its most rapid variation. Then we can adopt a profile which is analytically described as follows: let us define

$$H(x) := 1 + \tanh\left(\frac{\sigma + x}{\delta_{wh}}\right) \tanh\left(\frac{\sigma - x}{\delta_{bh}}\right), \quad (22)$$

where  $\delta_{wh}, \delta_{bh} > 0$  are length scales describing the “thickness” of the region over which a rapid variation of the refractive index occurs, and  $2\sigma > 0$  corresponds to the overall spatial extension of the RIP. Compare also the choice of the velocity profile for acoustic geometries chosen in [33]. Then for the refractive index profile we choose

$$n(x) = n_0 + \eta \frac{H(x)}{\max_x H(x)}. \quad (23)$$

It is evident that asymptotically  $n$  converges to  $n_0$  and the RIP is correctly normalized. This model can be also useful in view of exploring the possibility to obtain a bh-wh laser [33].

From a physical point of view, we point out that, in the presence of a shock front typical values of  $\delta_{wh}, \eta$  may be order of  $\delta_{wh} \sim 10^{-6} \text{m}$  and  $\eta \sim 5 \times 10^{-3}$  respectively, leading to  $T_+ \sim 1 \text{K}$ .

## III. QUANTUM FIELD THEORETICAL TREATMENT: HAWKING RADIATION IN THE LABORATORY FRAME

The deduction of Hawking radiation emission by a (non-extremal) black hole can also be carried out in a dynamical situation where the effects of the onset of a black hole horizon on quantum field theory modes is taken into account. See e.g. the seminal paper by Hawking [2] and also [34], where the equivalence between the static and the dynamical picture for Hawking radiation is considered. In the following section, we develop a dynamical picture for particle production, with the aim of not relying only on the analogue gravity picture but also obtaining the phenomenon in a nongeometrical setting by means of standard tools of quantum field theory.

Consider a massless scalar field propagating in a dielectric medium and satisfying Eq. (1). It is not difficult to show that this equation arises in the eikonal approximation for the components of the electric field perturbation in a nonlinear Kerr medium [12,15].  $n(x_l - vt_l)$  is the refractive index, which accounts for the propagating refractive index perturbation in the dielectric.

In order to develop the model we adopt the following strategy: a) we look first for a complete set of solutions of Eq. (1); b) then we employ these solutions to perform a comparison between an initial situation, consisting of an unperturbed dielectric without a laser pulse, with a uniform

and constant refractive index  $n_0$ , and a final situation where a laser pulse inducing a superluminal RIP is instead present. This scenario is analogous to the original situation envisaged and treated by Hawking [2] in which one considers an initial spherically symmetric astrophysical object, with no particles present at infinity, followed by a collapsing phase leading eventually to an evaporating Schwarzschild black hole. In analogy with the original treatment by Hawking, our aim is to evaluate the mean value of the number of quanta (calculated by means of final creation and annihilation operators) on the initial vacuum state.

### A. Out states

Recast Eq. (1) in terms of the new retarded and advanced variables, respectively,  $u = x_l - vt_l$  and  $w = x_l + vt_l$ :

$$\frac{n^2(u)v^2}{c^2}(\partial_u^2\Phi + \partial_w^2\Phi - 2\partial_u\partial_w\Phi) - (\partial_u^2\Phi + \partial_w^2\Phi + 2\partial_u\partial_w\Phi) - \partial_y^2\Phi - \partial_z^2\Phi = 0, \quad (24)$$

and look for monochromatic solutions of the form

$$\Phi(u, w, y, z) = A(u)e^{ik_w w + ik_y y + ik_z z}. \quad (25)$$

Then, the ansatz of Eq. (25) leads to

$$A''(u) + 2ik_w \frac{c^2 + n^2(u)v^2}{c^2 - n^2(u)v^2} A'(u) - \left(k_w^2 + \frac{k_y^2 + k_z^2}{1 - n^2(u)\frac{v^2}{c^2}}\right) A(u) = 0. \quad (26)$$

Assuming  $n(u)$  to be analytic (see e.g. (17)) we see that the coefficients of  $A'(u)$  and of  $A(u)$  have a first order pole at the roots of

$$1 - n(u)\frac{v}{c} = 0. \quad (27)$$

Then, assuming  $n$  to be of the form (4), the condition for the occurrence of the event horizons is given by (10). Since  $x = \gamma u$ , the black hole and white hole horizons, when they exist, are located, respectively, at

$$u_{\pm} = \frac{1}{\gamma} x_{\pm}, \quad (28)$$

and we have a second order linear differential equation with Fuchsian singular points at  $u = u_{\pm}$  (with  $u_- < u_+$ ). The general solution of Eq. (24) can be determined, in the neighborhood of the singular points  $u_{\pm}$ , by the standard methods of integration by series. We perform this calculation in Appendix C. Here, instead, since we are interested in the asymptotic behavior of the solutions as  $u \rightarrow u_{\pm}$  and as  $u \rightarrow \pm\infty$ , the best strategy is to appeal to the WKB approximation. Namely, we write  $\Phi$  as in Eq. (25) with

$$A(u) = \exp\left(i \int^u k_u(u') du'\right). \quad (29)$$

Then, we find the following dispersion relation:

$$\frac{n^2 v^2}{c^2} (k_u - k_w)^2 - (k_u + k_w)^2 - k_{\perp}^2 = 0, \quad (30)$$

where  $k_{\perp}^2 = k_y^2 + k_z^2$ . By solving for  $k_u$  as a function of  $k_w$ ,  $\mathbf{k}_{\perp}$  one arrives at a second degree equation whose solutions are of the form:

$$k_u^{\pm} = -\frac{k_w}{1 - \frac{n^2 v^2}{c^2}} \left[ 1 + \frac{n^2 v^2}{c^2} \pm 2 \frac{nv}{c} \sqrt{1 - \frac{k_{\perp}^2}{k_w^2} \frac{1 - \frac{n^2 v^2}{c^2}}{4 \frac{n^2 v^2}{c^2}}} \right]. \quad (31)$$

The constraint

$$\frac{k_{\perp}^2}{k_w^2} \leq \frac{4 \frac{n^2 v^2}{c^2}}{1 - \frac{n^2 v^2}{c^2}}, \quad (32)$$

must hold in order that propagating solutions are available. This seemingly would imply the existence of a limiting angle for the emission, but one has to take into consideration that, in approaching the horizon, the right-hand side of the latest equation becomes infinite, leaving room for no real limitation on the emission angle.

The root  $k_u^+$  is singular at the horizons, whereas  $k_u^-$  is regular there. We focus on the behavior of  $k_u^+$  in the neighborhood of  $u_+$ . Near  $u_+$  the dependence on  $\mathbf{k}_{\perp}$  is washed out, in agreement with the analysis in [5], and we have

$$k_u \sim -\frac{2k_w}{1 - n\frac{v}{c}}. \quad (33)$$

Then,

$$A(u) \sim \exp\left[i \frac{2ck_w}{vn'(u_+)} \log(u - u_+)\right]. \quad (34)$$

The logarithmic divergence in the phase as  $u \downarrow u_+$  is entirely analogous to the one experienced by the covariant wave equation at the horizon of a Schwarzschild black hole [2,3] and represents the typical behavior of outgoing modes approaching the horizon [5]. It is worth to remark that this divergence appears here without any reference to geometry. In addition, as shown in Appendix C, it emerges naturally as an exact asymptotic (as  $u \downarrow u_+$ ) consequence of Eq. (24). On the other hand, the same asymptotic behavior can be inferred also in the context of the geometric approach. Indeed, such behavior is characteristic also of the monochromatic solutions of the covariant wave equation in the metric (2), though the latter equation is not exactly solvable.

Regarding the dependence on  $\vec{k}_{\perp}$ , we note that it does not disappear far away from the horizon, but this turns out not to be relevant for the Hawking phenomenon [5]. Also note that  $k_u$  has the opposite sign of  $k_w$ , and  $|k_u| > |k_w|$ .

We denote by  $F^+$  the solutions associated to the root  $k_u^+$ . Their asymptotic form (at large distances from the pulse, i.e. for  $u \gg u_+$ ) is

$$F^+ \sim e^{ik_u u + ik_w w + ik_y y + ik_z z}, \quad (35)$$

or

$$F^+ \sim e^{i(k_u+k_w)x_l + ik_y y + ik_z z - i\omega_l t_l (k_u - k_w)}. \quad (36)$$

This can be written as

$$F^+ = F_{\mathbf{k}_l}^+ \sim \exp(i\mathbf{k}_l \cdot \mathbf{x}_l - i\omega_l t_l), \quad (37)$$

where  $\mathbf{k}_l = (k_{xl}, k_y, k_z)$ , with

$$\omega_l = v(k_u - k_w), \quad (38)$$

$$k_{xl} = k_u + k_w, \quad (39)$$

so that

$$k_u = \frac{1}{2} \left( k_{xl} + \frac{\omega_l}{v} \right), \quad (40)$$

$$k_w = \frac{1}{2} \left( k_{xl} - \frac{\omega_l}{v} \right). \quad (41)$$

We look for solutions which are asymptotically of positive frequency and outgoing with respect to the dielectric perturbation, which is right moving. In other words, we choose  $\omega_l > 0$  and  $k_{xl} > 0$ . Moreover, for definiteness we fix  $k_w < 0$ . As a consequence, (38) and (39) are satisfied for  $k_u > -k_w$ . As to  $\omega_l$ , it satisfies the asymptotic dispersion relation  $n_0^2 \omega_l^2 = \mathbf{k}_l^2 c^2$ . A similar analysis can be performed for the solution which is regular at  $u = u_+$ , which we denote by  $F_{\mathbf{k}_l, \text{reg}}^+$ . Similarly, we denote by  $F_{\mathbf{k}_l}^-$  and by  $F_{\mathbf{k}_l, \text{reg}}^-$  the singular and, respectively, regular solutions at  $u_-$ .

Introducing the angle of emission  $\theta$ , defined by  $k_x = |\mathbf{k}_l| \cos(\theta)$ , Eq. (41) can be written as

$$k_w = -\frac{\omega_l}{2v} \left( 1 - \frac{v}{c} n_0 \cos(\theta) \right). \quad (42)$$

### B. In states

Prior to the formation of the pulse the wave equation is

$$\frac{n_0^2}{c^2} \partial_{t_l}^2 \Phi - \partial_{x_l}^2 \Phi - \partial_y^2 \Phi - \partial_z^2 \Phi = 0, \quad (43)$$

whose independent positive frequency plane wave solutions are trivially given by (with  $\omega_l > 0$ )

$$F_{\mathbf{k}_l}^{\text{in}} = \exp(i\mathbf{k}_l \cdot \mathbf{x}_l - i\omega_l t_l), \quad (44)$$

where the obvious dispersion relation holds

$$\frac{n_0^2}{c^2} \omega_l^2 - k_{xl}^2 - k_\perp^2 = 0, \quad (45)$$

and where  $k_\perp^2 = k_y^2 + k_z^2$ .

### C. Quantization

We can separate the monochromatic solutions  $\{F^+, F_{\text{reg}}^+\}$  corresponding to all possible values of  $\mathbf{k}_l$ , into positive and

negative frequency components  $\{f_{\mathbf{k}_l}, f_{\mathbf{k}_l}^*\}$ . These form a complete set of solutions which are mutually orthogonal with respect to the index  $\mathbf{k}_l$  relative to the Klein-Gordon product in the retarded variables:

$$(\Psi, \Phi) = i \int_{\mathbb{R}^3} dudydz \left( \Psi^* \frac{\partial \Phi}{\partial w} - \Phi \frac{\partial \Psi^*}{\partial w} \right). \quad (46)$$

The above product is a function of  $w$  (due to the factor  $n^2(x_l - vt_l)$ ) it is indeed impossible to define in a standard way a conserved inner product for Eq. (1)). Expanding an arbitrary solution  $\Phi$  of Eq. (1) over the modes  $\{f_{\mathbf{k}_l}, f_{\mathbf{k}_l}^*\}$  we quantize the field by promoting  $\Phi$  to a field operator according to the expansion

$$\Phi = \int d^3 \mathbf{k}_l (a_{\mathbf{k}_l} f_{\mathbf{k}_l} + a_{\mathbf{k}_l}^* f_{\mathbf{k}_l}^*), \quad (47)$$

where the creation and annihilation operators  $a_{\mathbf{k}_l}$ ,  $a_{\mathbf{k}_l}^*$  satisfy the usual commutation relations

$$[a_{\mathbf{k}_l}, a_{\mathbf{k}_l'}^*] = \delta_{\mathbf{k}_l, \mathbf{k}_l'}. \quad (48)$$

Comparison between the creation and annihilation operators of the *in* and *out* states will allow us to calculate the Bogoliubov coefficients relating one set to the other, thus leading us to evaluate the average number of emitted quanta.

### D. Greybody factor

In our four dimensional problem, *a priori* we cannot neglect the effect of backscattering, i.e., the fact that, once emitted near the horizon, photons can undergo with a certain probability, reflection back into the horizon due to the presence of a nonvanishing potential that they encounter in their propagation [2]. This originates the so-called greybody factor, which can be calculated as the square modulus of the transmission coefficient.

Consider a massless scalar field propagating in a dielectric medium and satisfying the massless Klein-Gordon equation in the pulse metric (7), and let us call  $g(x)$  the component  $g_{\tau\tau}$  appearing in (8). We get

$$\frac{n^2(x)}{c^2} \frac{1}{g(x)} \partial_\tau^2 \Phi - n(x) \partial_x \left( \frac{1}{n(x)} g(x) \partial_x \Phi \right) - \partial_y^2 \Phi - \partial_z^2 \Phi = 0. \quad (49)$$

We consider solutions of the form

$$\Phi(\tau, x, y, z) = e^{i\omega\tau} \varphi(x) e^{-ik_y y - ik_z z}. \quad (50)$$

As a consequence, we obtain an equation of the form

$$(p(x)\varphi')' + \omega^2 k(x)\varphi - q(x)\varphi = 0, \quad (51)$$

where the prime stays for the derivative with respect to  $x$  and

$$p(x) := \frac{g}{n}; \quad k(x) := \frac{n}{c^2 g}; \quad q(x) := k_\perp^2 \frac{1}{n}. \quad (52)$$

The following change of variable:

$$s(x) := \int^x du \sqrt{\frac{k(u)}{p(u)}} = \int^x du \frac{n}{cg}, \quad (53)$$

where  $s$  results to coincide with the tortoise coordinate, leads to the following Schrodinger-like form of the equation for  $\varphi$ :

$$\frac{d^2 \varphi}{ds^2} + (\omega^2 - Q(s))\varphi = 0, \quad (54)$$

where

$$Q(s) := k_{\perp}^2 \frac{c^2 g}{n}. \quad (55)$$

It can be noticed that, as  $x \rightarrow x_+$ , i.e. as  $s \rightarrow -\infty$ , one obtains  $Q(s) \rightarrow 0$ , and that

$$\lim_{s \rightarrow \infty} Q(s) = k_{\perp}^2 \frac{c^2}{n_0^2} \left(1 - n_0^2 \frac{v^2}{c^2}\right) =: Q_{\infty}. \quad (56)$$

$Q(s)$  turns out to be monotonically increasing from 0 to the aforementioned constant value as  $s \rightarrow \infty$ , and the transition from the zero value and the asymptotically constant value for a bump is very fast, in such a way that, as a plausible approximation for the calculation of the transmission coefficient, we replace the above potential  $Q(s)$  with a steplike effective potential with height  $Q_{\infty}$ :

$$\frac{d^2 \varphi}{ds^2} + (\omega^2 - Q_{\infty})\varphi = 0. \quad (57)$$

In this approximation, we can refer to well-known results of standard quantum mechanics (see e.g. [35], where the step barrier appears as a subcase of problem 37), in order to infer that:

(a) the transmission coefficient is zero if

$$\omega < \sqrt{Q_{\infty}} = k_{\perp} \frac{c}{n_0} \sqrt{\left(1 - n_0^2 \frac{v^2}{c^2}\right)}. \quad (58)$$

(b) the transmission coefficient is

$$|T|^2 = 4 \frac{\omega \sqrt{Q_{\infty}}}{(\omega + \sqrt{Q_{\infty}})^2} = \Gamma, \quad (59)$$

for

$$\omega > \sqrt{Q_{\infty}}. \quad (60)$$

We point out that the dispersion relation

$$k_x^2 + k_{\perp}^2 = \frac{n_0^2}{c^2} \omega^2 \quad (61)$$

implies that a) cannot occur. As a consequence, the Heaviside function which, in principle, should multiply the aforementioned  $|T|^2$ , is always equal to 1.

We have to translate the latter result in the laboratory frame; the simple substitution

$$\omega = (\omega_l - vk_{xl})\gamma, \quad (62)$$

where the label  $l$  indicates laboratory frame quantities, leads to the following expression for the greybody factor:

$$\Gamma(\omega_l, \mathbf{k}_{\perp}) \simeq \frac{4(\omega_l - vk_{xl})\sqrt{(\omega_l - vk_{xl})^2 - k_{\perp}^2 q_0^2}}{[(\omega_l - vk_{xl}) + k_{\perp} q_0]^2}, \quad (63)$$

where

$$q_0 = \frac{c}{n_0} \sqrt{1 - n_0^2 \frac{v^2}{c^2}}. \quad (64)$$

One can also take into account that

$$\omega_l - vk_{xl} = \omega_l \left(1 - n_0 \frac{v}{c} \cos(\theta)\right), \quad (65)$$

so that

$$\begin{aligned} \Gamma &\simeq \frac{4(1 - n_0 \frac{v}{c} \cos(\theta)) |n_0 \frac{v}{c} - \cos(\theta)|}{[1 - n_0 \frac{v}{c} \cos(\theta) + |n_0 \frac{v}{c} - \cos(\theta)|]^2} \\ &\times \theta_H\left(\frac{\omega}{q_0} (1 - n_0 \frac{v}{c} \cos(\theta)) - k_{\perp}\right). \end{aligned} \quad (66)$$

Being  $\eta \ll 1$ , and  $v/c = (n_0 + k\eta)^{-1}$ ,  $k \in (0, 1)$ , we obtain

$$1 - n_0 \frac{v}{c} \cos(\theta) \sim 1 - \cos(\theta) + \frac{k\eta}{n_0} \cos(\theta), \quad (67)$$

$$n_0 \frac{v}{c} - \cos(\theta) \sim 1 - \cos(\theta) - \frac{k\eta}{n_0}, \quad (68)$$

which leads to the conclusion that

$$\Gamma \simeq 1. \quad (69)$$

It is worth pointing out that, on the grounds of the previous result, a 90-degree emission would not suffer any substantial suppression by backscattering. This is particularly relevant in connection to recent measurements of 90-degree photon emission from RIP-induced horizon [26].

## E. Thermal spectrum

In analogy with the corresponding black hole calculations, in order to find the evaporation temperature of the RIP we must compute the expectation value of the number operator of the outgoing photons evaluated in the *in* vacuum. In this connection, it is important to remark that, contrary to the standard black hole scenario, in the case of a RIP propagating in a nonlinear Kerr medium we have, strictly speaking, no real collapse situation. Nevertheless, we can “simulate a collapse” by building up a refractive index perturbation starting from an initial situation in which no signal is present in the dielectric.



We perform a coordinate shift which moves the black hole horizon in  $u = 0$ . A photon starting at  $u < 0$  is trapped and cannot reach the front observer. As a consequence, states with  $u < 0$  cannot be available to the front observer and he will be led to consider in his description only the  $u > 0$  part of the  $F^+$  solution, thus multiplying the latter by a Heaviside function  $\theta(u)$ . The Bogoliubov coefficients  $\alpha_{\mathbf{k},\mathbf{k}'}$  and  $\beta_{\mathbf{k},\mathbf{k}'}$  relating, respectively, the positive and negative frequency components between the initial *in* and final *out* states are analyzed in D, and satisfy the fundamental relation

$$\sum_{\mathbf{k}'} |\alpha_{\mathbf{k},\mathbf{k}'}|^2 = e^{(4\pi c k_w / n'(u_+)v)} \sum_{\mathbf{k}'} |\beta_{\mathbf{k},\mathbf{k}'}|^2, \quad (70)$$

which is also proved in the Appendix. Moreover, the expectation value of the number operator of the outgoing photons in the initial vacuum state is

$$\langle 0 | n | N_{\mathbf{k}_l}^{out} | 0 \rangle = \sum_{\mathbf{k}'} |\beta_{\mathbf{k},\mathbf{k}'}|^2. \quad (71)$$

Then, by using the completeness relation for the Bogoliubov coefficients and taking into account backscattering,

$$\sum_{\mathbf{k}'} (|\alpha_{\mathbf{k},\mathbf{k}'}|^2 - |\beta_{\mathbf{k},\mathbf{k}'}|^2) = \Gamma(\omega_l, \mathbf{k}_\perp), \quad (72)$$

and Eq. (70), we obtain the following thermal-like distribution written in terms of asymptotic (physical) frequencies:

$$\langle N_{\mathbf{k}_l} \rangle = \frac{\Gamma}{\exp(\frac{\hbar\omega_l}{k_b T}) - 1} \quad (73)$$

where

$$T = v^2 \frac{\hbar}{2\pi k_B c} \frac{1}{1 - \frac{v}{c} n_0 \cos\theta} \left| \frac{dn}{du}(u_+) \right|. \quad (74)$$

By comparing this temperature with the temperature  $T_+$  derived in the analogue Hawking model in the pulse frame (see Eq. (13)), we find the relation

$$T = \frac{1}{\gamma} \frac{1}{1 - \frac{v}{c} n_0 \cos(\theta)} T_+. \quad (75)$$

For the same values of the parameters leading to Eq. (21), for  $\theta = 0$  and for a typical value  $v \sim \frac{2}{3}c$ , we would obtain

$$T \sim 78K, \quad (76)$$

which is again much greater than the values of  $T$  for typical acoustic black holes. Moreover, in the specific case of a shock front, this temperature may increase significantly so that  $T \sim 2000K$  (cf. Subsection II C).

Equation (75) gives the correct transformation law for the temperature in going over from the pulse frame to the laboratory frame. Indeed, to find how the temperature transforms, start from Wien's displacement law which

gives the wavelength of maximum emission of a blackbody as a function of the temperature

$$\lambda_{\max} T = b = 2.9 \times 10^{-3} m \times K. \quad (77)$$

Converting to the frequency of maximum emission gives

$$\omega_{l\max} = \frac{2\pi c}{b} T. \quad (78)$$

Now, under the boost connecting the lab frame to the pulse frame, the frequency transforms according to the relativistic Doppler formula in a medium with refractive index  $n_0$ ,

$$\omega = \omega_l \gamma \left( 1 - \frac{v}{c} n_0 \cos\theta \right), \quad (79)$$

where  $\omega$  is the frequency in the pulse frame, and  $\theta$  is the emission angle with respect to the direction of motion of the pulse in the lab frame. Combining Eq. (79) with Eq. (78) gives Eq. (75).

#### IV. EFFECTS OF OPTICAL DISPERSION

In the framework of analogue models for black holes (dumb holes, bec models) there is a number of studies devoted to the analysis of the actual generation of Hawking radiation (see e.g. [9,33,36–43]). In particular, a suitable dispersion law, which involves the fourth power of the momentum  $k$  in the comoving frame is considered, since the original model by Unruh [36]. Calculations, in particular, in presence of a black hole—white hole system, are quite involved and require also numerical simulations.

We do not deal herein quantitatively with the problem of mode conversion, and we do not consider the problem of black hole lasers (see [9,33,36–44]), which will be taken into account in further developments of the present model. Still, we aim to point out the main physical consequences due to optical dispersion, and we shall stress that optical dispersion behaves as a fundamental ingredient for dielectric models. In this respect, there is agreement with dispersive models in sonic black holes.

We start by recalling how optical dispersion affects the dispersion relation for the quantum electromagnetic field in linear homogeneous dielectric media (see e.g. [45]):

$$n^2(\omega_l) \omega_l^2 - \mathbf{k}_l^2 c^2 = 0. \quad (80)$$

This dispersion relation is obeyed by the monochromatic components of the field, and in [45] phenomenological quantization of the electromagnetic field in a dielectric medium is justified on the grounds of a more rigorous approach. In the presence of optical dispersion

$$\Phi(\mathbf{x}_l, t_l) = \int_0^\infty d\omega_l \phi(\omega_l, \mathbf{x}_l, t_l), \quad (81)$$

where  $\phi(\omega_l, \mathbf{x}_l, t_l) = \phi_+(\omega_l, \mathbf{x}_l) e^{-i\omega_l t_l} + \phi_-(\omega_l, \mathbf{x}_l) e^{i\omega_l t_l}$  are the monochromatic components. Then it is easy to show that, even by allowing a spatial dependence for the refractive index

$$\nabla \varphi(\omega_l, \mathbf{x}_l) + \omega_l^2 \frac{n^2(\omega_l, \mathbf{x}_l)}{c^2} \varphi(\omega_l, \mathbf{x}_l) = 0. \quad (82)$$

Moreover, in the case of an homogeneous medium, by letting  $\varphi(\omega, \mathbf{x}) \propto \exp(i\mathbf{k}_l \cdot \mathbf{x})$ , one finds that Eq. (82) is satisfied only if

$$|\mathbf{k}_l| = \omega_l \frac{n(\omega_l)}{c}, \quad (83)$$

i.e. only if each monochromatic component travels at its phase velocity (as obvious).

As a typical example, in a so-called 1-resonance model, we get the Sellmeier equation

$$n^2(\omega_l) = 1 + \frac{\omega_c^2}{\omega_0^2 - \omega_l^2}, \quad (84)$$

where  $\omega_0$  is the resonance frequency and  $\omega_c$  is the coupling constant, which is also called plasma frequency. It is worth pointing out that the function  $n^2(\omega_l)$  is a Lorentz scalar and it is invariant under boosts [it may be written in a manifestly covariant form as  $n^2(v^\mu k_\mu)$ , where  $v^\mu$  stays for the velocity 4-vector and  $k^\mu$  is the four-momentum whose zeroth covariant component in the lab coincides with  $\omega_l$ ; cf. [46]]. For the case of interest in Ref. [26], which concerns optical dispersion in fused silica, it can be shown that a more suitable formula involves the summation of three terms like the frequency-dependent ones appearing on the right-hand side of (84). The refractive index arising from the complete Sellmeier for the case discussed in Ref. [26] is displayed in Subsection IV B. Still, we note that, although complete expressions can be provided for the dispersion relation, for practical purposes it is very useful to use the so-called Cauchy formula

$$n(\omega_l) = n_0 + B_0 \omega_l^2, \quad (85)$$

where  $n_0$  is the would-be refractive index in absence of optical dispersion and  $B_0$  is a suitable constant; this approximation works well for fused silica in the visible and near-infrared (below 1  $\mu\text{m}$  wavelengths) spectral region where it is practically indistinguishable from the full Sellmeier relation. Differences may become more evident between the Cauchy and Sellmeier relations if one attempts to calculate the gradients of the curves, e.g. in order to evaluate the group index or group velocity in the medium. However, once the RIP velocity is fixed we are only concerned with the phase velocities and the Cauchy formula approximation is sufficiently precise.

### A. Dispersion and RIP

We note that the dispersion relation for a linear homogeneous dielectric medium is quite different from the one which has been hitherto considered in the sonic model by Unruh and its subsequent developments (see also analogous comments in Ref. [47]).

We may also wonder if the above relation can be trusted in our case, as we started from a nonlinear medium which is also nonhomogeneous because of the RIP. We recall that we assume to be involved with linearized quantum fields around our effective geometry, and then it is reasonable to explore what happens in the presence of optical dispersion by considering linear dispersion effects. Moreover, we make the following ansatz:

$$n(x_l - vt_l, \omega_l) = n_0(\omega_l) + \eta f(x_l - vt_l), \quad (86)$$

i.e. we neglect optical dispersion effects in the RIP, and keep trace of it only in the background value  $n_0$ , which does not depend on spacetime variables, and passes from the status of constant to that of a function depending on  $\omega_l$ .

In order to justify *a posteriori* this ansatz we can take into account what happens in the usual modelization of dielectric media when inhomogeneities are considered. A possibility is to account for inhomogeneities by including space-dependent density, polarizability, resonant frequencies, and so on (see e.g. [48]). By neglecting dissipation in a polariton model, one can obtain a dielectric susceptibility which depends on space and frequency, in such a way that the Sellmeier formula changes only because of an explicit dependence of  $\omega_c$ ,  $\omega_0$  on space variables. In our case, because of the properties of fused silica in the visible region, where the Cauchy formula works well, we can even neglect the spatial dependence of  $\omega_0$  (which should be considered much greater than  $\omega_l$  and  $\omega_c$ ), and then, recalling that  $u := x_l - vt_l$ , on the grounds of Ref. [48] we assume that

$$\omega_c^2(u) = \omega_{c0}^2 + \delta \omega_{c1}^2(u), \quad (87)$$

where  $\delta \ll 1$  is a small parameter; if we also define

$$n_0(\omega_l) = \sqrt{1 + \frac{\omega_{c0}^2}{\omega_0^2 - \omega_l^2}}, \quad (88)$$

the Sellmeier Eq. (84) then leads to

$$\begin{aligned} n(\omega_l, u) &= n_0(\omega_l) \sqrt{1 + \delta \frac{1}{n_0^2(\omega_l)} \frac{\omega_{c1}^2(u)}{\omega_0^2 - \omega_l^2}} \sim n_0(\omega_l) \\ &+ \frac{\delta}{2n_0 \omega_0^2} \omega_{c1}^2(u), \end{aligned} \quad (89)$$

where we have used the Cauchy approximation and neglected terms order of  $\delta \omega_l^2$ . This validates our ansatz (86), with straightforward identifications. Another possibility is to adopt in the visible frequency region the following perturbative ansatz:

$$\begin{aligned} n(\omega_l, u) &= n_0 + \eta A_1(u) + (B_0 + \eta B_1(u)) \omega_l^2 \sim n_0 \\ &+ B_0 \omega_l^2 + \eta A_1(u), \end{aligned} \quad (90)$$

where  $n_0 + B_0 \omega_l^2 \equiv n_0(\omega_l)$  and where we neglect terms  $O(\eta \omega_l^2)$ . The latter approximation is very useful, because we can write

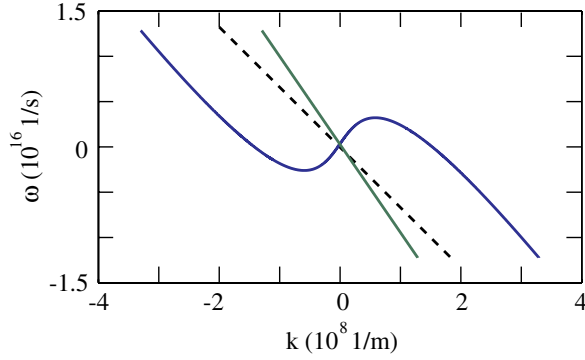


FIG. 2 (color online). Branches relative to the dispersion relation (94) in the comoving frame. The dashed line corresponds to  $\omega = -vk_x$ . Comoving frequencies corresponding to the experiments in Ref. [26] are of the order of  $10^{13}$  rads/s.

$$n(\omega_l, u) = n_0(\omega_l) + \eta f(u) = n(u) + B_0 \omega_l^2, \quad (91)$$

which isolates the contribution of the optical dispersion. As for the dispersion relation, we obtain in the lab

$$n(\omega_l, u) \omega_l = \pm |\mathbf{k}_l| c, \quad (92)$$

i.e.

$$n(u) \omega_l + B_0 \omega_l^3 \pm |\mathbf{k}_l| c = 0. \quad (93)$$

This can be carried into the comoving (pulse) frame, where one obtains

$$n(x) \gamma (\omega + vk_x) \pm \sqrt{\gamma^2 \left( k_x + \frac{v}{c^2} \omega \right)^2 + k_\perp^2} c + B_0 \gamma^3 (\omega + vk_x)^3 = 0, \quad (94)$$

i.e. the same dispersion relation as in the absence of optical dispersion except for the cubic term  $\propto B_0$ . We shall consider in what follows the 2D reduction, where  $k_\perp = 0$ . Graphical solutions of the above equation are displayed in Fig. 2. Note that

$$\omega_l = \gamma (\omega + vk_x), \quad (95)$$

and then positive frequencies in the lab correspond to the region above the dashed line

$$\omega = -vk_x. \quad (96)$$

## B. Dispersion and horizons

The definition of the horizon which is suitable for this kind of situation, where one cannot recover a metric by reinterpreting the dispersion relation as an eikonal approximation of the wave equation (due to the presence of more than quadratic terms), can *a priori* involve phase velocity and/or group velocity; cf. e.g. a short discussion in [49]. Let us first consider what happens when one involves the phase velocity, indicated as  $v_{l,\varphi}$  in the lab reference frame and as  $v_\varphi$  in the comoving frame:

$$v_{l,\varphi} = v \Leftrightarrow v_\varphi = 0. \quad (97)$$

In the lab frame the calculation is elementary, and leads to

$$v_{l,\varphi} = \pm \frac{c}{n(\omega_l, u)}, \quad (98)$$

which, due to (97), implies the horizon condition

$$n(\omega_l, u) = \frac{c}{v}; \quad (99)$$

it is worth noticing that condition (99) is a straightforward generalization to the dispersive case of the condition obtained in the absence of dispersion. It is also important to point out that the horizon condition in the lab in presence of dispersion can be also expressed as follows:

$$1 - n(\omega_l, u) \frac{v}{c} = 0, \quad (100)$$

which e.g. in the Gaussian model provides

$$u_\pm = \pm \frac{\sigma}{\gamma} \sqrt{-2 \log \left[ \left( \frac{c}{v} - n_0(\omega_l) \right) \frac{1}{\eta} \right]}, \quad (101)$$

and, in general, a modified horizon condition appears:

$$n_0(\omega_l) < \frac{c}{v} \leq n_0(\omega_l) + \eta. \quad (102)$$

This condition is displayed in Fig. 3 (where wavelength replaces  $\omega_l$ ) and the velocity of the RIP is chosen to match the experimental conditions of Ref. [26]. Explicitly, by using the Cauchy formula (85), we obtain (we consider only positive frequencies) for  $n_0 + \eta < \frac{c}{v}$ , i.e. for a relatively small perturbation (see e.g. [26]):

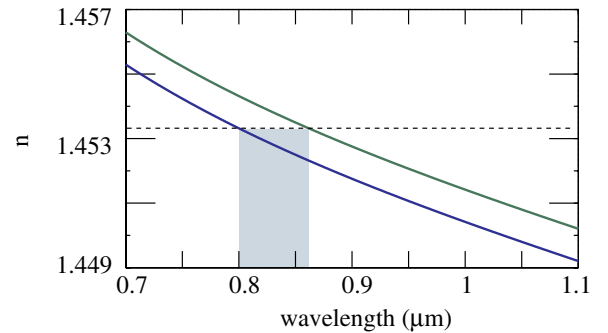


FIG. 3 (color online). Prediction of the Hawking emission spectral range accounting for the RIP velocity and the material refractive index  $n$  from a complete Sellmeier equation. The two curves show, for the case of fused silica,  $n_0$  (in blue) and  $n_0 + \delta n$  (in green) with  $\delta n = 1 \times 10^{-3}$ . The shaded area indicates the spectral emission region predicted for a Bessel filament that has an effective refractive index  $n = c/v_B$  where the Bessel peak velocity is  $v_B = v_G / \cos \theta$  with cone angle  $\theta = 7$  deg and  $v_G = d\omega/dk$  is the usual group velocity. The values and parameters used correspond to the experiments shown in Ref. [26].

$$\frac{\sqrt{\frac{c}{v} - n_0} - \eta}{\sqrt{B_0}} \leq \omega_l < \frac{\sqrt{\frac{c}{v} - n_0}}{\sqrt{B_0}}, \quad (103)$$

i.e., in terms of wavelength,

$$\frac{2\pi c \sqrt{B_0}}{\sqrt{\frac{c}{v} - n_0}} < \lambda \leq \frac{2\pi c \sqrt{B_0}}{\sqrt{\frac{c}{v} - n_0} - \eta}. \quad (104)$$

If, instead,  $n_0 + \eta \geq \frac{c}{v}$ , i.e. for a substantially larger perturbation, we obtain

$$0 \leq \omega_l < \frac{\sqrt{\frac{c}{v} - n_0}}{\sqrt{B_0}}, \quad (105)$$

i.e., in terms of wavelength,

$$\frac{2\pi c \sqrt{B_0}}{\sqrt{\frac{c}{v} - n_0}} < \lambda < \infty. \quad (106)$$

For the parameters of Fig. 3 (see caption), Eq. (105) is verified for  $\eta > 0.013$ , a rather large yet not completely unrealistic refractive index variation. We may evaluate the influence of material dispersion in relation to the event horizon condition. In the absence of dispersion a horizon is created only if  $v$  is tuned with extreme care such that Eq. (102) with dispersionless  $n_0$  is satisfied. Bearing in mind the typically small values of  $\delta n \sim 10^{-3} - 10^{-4}$  (e.g. in fused silica,  $n_2 \sim 3 \times 10^{-16} \text{ cm}^2/\text{W}$  and  $I \sim 10^{13} \text{ W}/\text{cm}^2$ ), this would be no minor feat.

Note that the Hawking photons will therefore in general be emitted only in a bounded spectral window. This is somewhat different from the dispersionless case in which, once  $v$  is properly tuned so as to achieve the horizon condition, all frequencies are simultaneously excited, and, *a posteriori*, it can also justify why one can limit oneself to adopt the Cauchy formula instead of a more complete Sellmeier. The aforementioned finite spectral window for photon emission in turn implies that in the dispersionless case one should expect to observe the complete blackbody spectrum predicted by Hawking. Conversely, in the presence of dispersion in our analogue model, only a limited spectral region is excited and the blackbody spectral shape will not be discernible. Moreover, one has to take into account that, even in homogeneous transparent dielectrics, optical dispersion affects the spectral density of photons which is actually to be taken into account, by introducing a phase space factor multiplying the standard Planckian distribution term which depends on both the refractive index  $n(\omega)$  and on the group velocity  $v_g(\omega)$  [50]:

$$\rho(\omega) = \frac{\left(\frac{\hbar \omega^3}{\pi^2 c^2}\right) \frac{n^2(\omega)}{v_g(\omega)}}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1}. \quad (107)$$

This formula is intended to hold true in the reference of the thermal bath. These optical dispersion contributions amounts to a sort of greybody factor arising from the

interaction of photons with the dielectric material. See e.g. [50]. This is *per se* a sufficient reason for expecting deviations from the standard Planckian distribution. A very naive inclusion of dispersion in our case would also lead to  $T = T(\omega)$ , which would make even less plausible a pure Planckian spectrum.

Regarding the width of the spectral emission window we note that this is determined by the value of  $\eta$ . For a typical case we consider a Gaussian pump pulse in fused silica,  $n_g(800 \text{ nm}) = 1.467$  and we find that for  $\delta n = 10^{-3}$ , Eq. (102) is satisfied over a  $\sim 15 \text{ nm}$  bandwidth at  $435 \text{ nm}$ . This window may become substantially large in lower dispersion regions as shown in Fig. 3.

The phase velocity horizon condition may also be calculated in the comoving frame, where one obtains  $v_\phi = 0$  if  $\omega = 0$  (for  $k_x \neq 0$ ), i.e. for  $v > 0$ :

$$n(\gamma(\omega + vk_x), x)|_{\{\omega(k_x, x)=0, k_x \neq 0\}} = \frac{c}{v}. \quad (108)$$

Notice that  $\omega = \omega(k_x, x)$  is solution of the 2D reduction of Eq. (94). The above condition requires that, as in absence of dispersion,  $n_0 < \frac{c}{v}$ . Moreover, if also  $n_0 + \eta < \frac{c}{v}$ , then two disconnected regions are obtained:

$$\begin{aligned} -\frac{\sqrt{\frac{c}{v} - n_0}}{\gamma v \sqrt{B_0}} < k_x &\leq -\frac{\sqrt{\frac{c}{v} - n_0 - \eta}}{\gamma v \sqrt{B_0}} \\ \frac{\sqrt{\frac{c}{v} - n_0 - \eta}}{\gamma v \sqrt{B_0}} &\leq k_x < \frac{\sqrt{\frac{c}{v} - n_0}}{\gamma v \sqrt{B_0}}, \end{aligned} \quad (109)$$

where only the latter region corresponds to  $\omega_l > 0$ . These are shown in Fig. 4. Of course, the latter condition is equivalent to condition (103) in the lab frame.

If, instead,  $n_0 + \eta \geq \frac{c}{v}$ , then a unique connected region is obtained, with

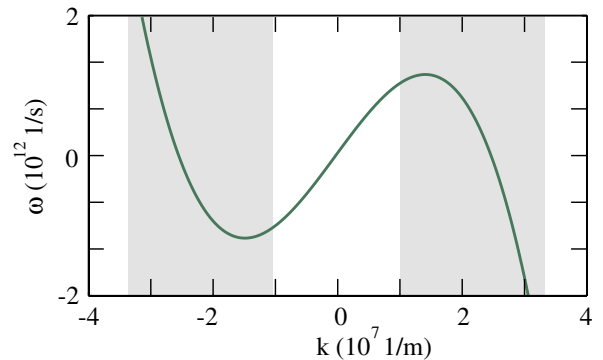


FIG. 4 (color online). The dispersion relation (96) in the comoving frame for a 2D model at a fixed point  $x$  (only the branch relevant to horizon formation is shown). The shaded regions delimit the bands representing the values of  $k_x$  for which a phase horizon exists. See Eqs. (110) and (111), in which we chose values of the velocity  $v$  and  $\eta$  for illustrative purposes and that are not related to the experimental values used in Fig. 3.



$$-\frac{\sqrt{\frac{c}{v}-n_0}}{\gamma v \sqrt{B_0}} < k_x < \frac{\sqrt{\frac{c}{v}-n_0}}{\gamma v \sqrt{B_0}}. \quad (110)$$

This condition is consistent with (105). For a further comment, see below.

One could also *a priori* consider a different definition of horizon, involving the group velocity instead of the phase velocity:

$$v_{l,g} = v \Leftrightarrow v_g = 0, \quad (111)$$

where the first equality holds in the lab and the second in the pulse frame. For simplicity we consider a 2D reduction of our model, so that  $k_\perp = 0$ . One obtains

$$v_{l,g} = v \Leftrightarrow \frac{c}{v} = n(\omega_l, u) + \frac{\partial n(\omega_l, u)}{\partial \omega_l} \omega_l. \quad (112)$$

In the case of the Cauchy formula, we find

$$\frac{c}{v} = n(\omega_l, u) + 2B_0 \omega_l^2 \equiv n_g(\omega_l, u), \quad (113)$$

where  $n_g$  indicates the group refractive index. In this case, in place of (102), one would obtain

$$n_g(\omega_l) < \frac{c}{v} \leq n_g(\omega_l) + \eta, \quad (114)$$

which presents a clear shift with respect to the window corresponding to (102). Explicitly, by referring again to (85), we find

$$\frac{\sqrt{\frac{c}{v}-n_0}-\eta}{\sqrt{3B_0}} \leq \omega_l < \frac{\sqrt{\frac{c}{v}-n_0}}{\sqrt{3B_0}}, \quad (115)$$

i.e., in terms of wavelength,

$$\frac{2\pi c \sqrt{3B_0}}{\sqrt{\frac{c}{v}-n_0}} < \lambda \leq \frac{2\pi c \sqrt{3B_0}}{\sqrt{\frac{c}{v}-n_0}-\eta}. \quad (116)$$

It is easy to realize that coexistence of phase and group horizons is allowed only if

$$\frac{\sqrt{\frac{c}{v}-n_0}}{\sqrt{3B_0}} \geq \frac{\sqrt{\frac{c}{v}-n_0}-\eta}{\sqrt{B_0}}, \quad (117)$$

i.e. if

$$\eta \geq \frac{2}{3} \left( \frac{c}{v} - n_0 \right). \quad (118)$$

Note that experiments, at least if the latter condition is not satisfied, should be able to distinguish between the two horizon conditions for phase velocity and group velocity given above. For example, in the experimental conditions of Ref. [26] used also in Fig. 3, condition (118) implies  $\eta > 0.009$ . However the maximum  $\eta$  obtained was only  $\sim 0.001$  and indeed the experiments clearly reveal emission in correspondence to the *phase* horizon alone.

It is possible to show that, if  $\eta > \frac{c}{v} - n_0$ , there is a region in the  $(\omega, k_x, x)$  space where no group horizon appears because of the lack of any real zero in the derivative of the dispersion equation. In order to realize this fact analytically, we can proceed as follows. Let us consider in

the comoving frame the dispersion equation in the following form:

$$D = D_+ D_- = 0, \quad (119)$$

where

$$D_\pm := n(x) \gamma (\omega + v k_x) + B_0 \gamma^3 (\omega + v k_x)^3 \pm \gamma \left( k_x + \frac{v}{c^2} \omega \right) c = 0. \quad (120)$$

The dispersion relation  $\omega(k_x, x)$  solves  $D = 0$ . The only dispersion equation which can lead to a group velocity horizon is the solution  $\omega_-(k_x, x)$  of  $D_- = 0$ . Instead, the other branch  $\omega_+(k_x, x)$  which solves  $D_+ = 0$  is monotonically decreasing. We find

$$v_g = \frac{\partial \omega(k_x, x)}{\partial k_x} = \frac{\partial D}{\partial k_x} \left( \frac{\partial D}{\partial \omega} \right)^{-1}. \quad (121)$$

As a consequence, a group velocity horizon can emerge only by solving the system  $D_- = 0$  and  $\frac{\partial D_-}{\partial k_x} = 0$ . It is easy to show that  $\frac{\partial D_-}{\partial k_x} = 0$  leads to

$$k_x^{(-)} = -\frac{\omega}{v} \pm \frac{1}{\gamma v \sqrt{3B_0}} \sqrt{\frac{c}{v} - n(x)}. \quad (122)$$

It is immediate to see that, if  $n_0 + \eta \geq \frac{c}{v}$ , as in the case where a unique connected region is found for the phase velocity horizons (cf. Eq. (110)),  $k_x^{(-)}$  in (122) is complex valued for all  $x$  such that  $n(x) > \frac{c}{v}$ , for which then group velocity horizons disappear.

We may also qualitatively describe the behavior of a monochromatic or quasimonochromatic wave near a phase horizon. In the comoving frame, a horizon is approached only by waves that travel with  $v_\varphi > 0$ . Let us consider a wave traveling towards the white hole horizon; as it reaches the RIP, the refractive index increases, and then it slows down. This effect is enhanced by dispersion, because it implies a further increase of the refractive index. As the wave is as near as possible to the phase (white hole) horizon, it nearly stops, but, instead of stopping indefinitely, it suffers reflection (mode conversion) because of dispersion effects. This happens for all frequencies belonging to the allowed windows indicated above. The previous analysis is corroborated by the study of null geodesics, in the presence of optical dispersion, which is carried out in [46]. Indeed, therein it is shown (in a more general geometric setting) that only geodesics with  $v_\varphi \geq 0$  undergo a slowing down process but, instead of suffering a process of trapping, as happens in absence of optical dispersion, they are reflected away from the trapping point.

In the case of a wave packet, the dynamics appear to be more complex, but, due to its being a superposition of monochromatic components, we can infer that what happens should be a sort of “remastering” of the wave packet by the horizon in the following sense: if all of the



TABLE I. Comparison between the analogue Hawking and QFT descriptions of quantum vacuum emission by a moving RIP.

Black Hole—Pulse Frame	Quantum Field Theory—Lab Frame
propagation equation (eikonal approximation)	propagation equation (eikonal approximation)
↓	↓
appearance of a bh horizon	onset of the surface s.t. $1 - n(u)\frac{v}{c} = 0$
↓	↓
logarithmic singularity in the phase of quantum modes	logarithmic singularity in the phase of quantum modes
↓	↓
thermal spectrum with temperature $T_+$	thermal spectrum with temperature $T$

frequencies that form the wave packet fall within the phase horizon frequency window, then all of these will be reflected from the RIP. Each frequency will be reflected from a different point within the RIP (each corresponding to the horizon at the relative frequency) and this may severely distort the wave packet during reflection. Nevertheless, it is clear that if *all* spectral components are 100% reflected then there will be zero transmission. What is expected is that, in general, an incoming wave packet is converted into a reflected and possibly very broad and irregular wave train. Clearly, if part of the wave packet spectrum falls outside the phase horizon window, then these frequencies will be transmitted through the RIP and we can no longer talk of perfect blocking.

The action of a group velocity horizon appears to be different, in the sense that, although the existence of a window of allowed frequencies seems to be analogous to the one of a phase velocity horizon, we have to point out that in the present case  $\omega_l$  refers to the carrier wave in the wave train. The group velocity horizon appears to be less selective than the phase velocity one: it limits itself to cause a reflection of wave packets which have carrier frequencies in the allowed window, without distinguishing between monochromatic components composing the packet itself. The outgoing wave train is expected to be still in the form of a compact and relatively undisturbed wave packet. Both numerical simulations and experimental results could be able to reveal this different character of the two above horizon versions, and, in particular, measurements should be able to discern which definition is relevant for the physics at hand. It is worth pointing out that, in Ref. [26], the main role in photon production appears to be related to the phase velocity horizon rather than to the group velocity horizon.

## V. CONCLUSIONS

We have studied the photon production induced in a dielectric medium by a refractive index perturbation, which is created in a nonlinear medium by a laser pulse through the Kerr effect. The pulse has been assumed to have constant velocity  $v$ . In the pulse frame, we have investigated the analogous metric, and we have shown that two Killing horizons appear (black hole and white

hole) for a generic but static dielectric perturbation. Despite the fact that the analogue metric is determined up to an overall conformal factor, the temperature is conformally invariant. Then we have taken into account quantum field theory in the lab frame, in a way that is independent of the analogous geometric framework we have again shown that there is a photon production with a thermal spectrum, corresponding to the Hawking effect. These findings are summarized in Table I. We have also provided the transformation law of the temperature between the given frames and taken into account the effects of dispersion on the horizon condition.

## ACKNOWLEDGMENTS

F. B. wishes to thank Sergio Serapioni and Lesaffre Italia S.p.A. for financial support to the Department of Mathematics of the Università degli Studi di Milano, where part of this work was performed.

## APPENDIX A: FORMAL MAPPING TO AN ACOUSTIC BLACK HOLE

It is interesting to notice that our black hole metric in the pulse frame can be mapped into a form of acoustic black hole metric, provided that suitable identifications are made. In particular, we refer to the acoustic black hole metrics taken into consideration in [51], and limit ourselves to the analysis of the  $x - t$  part of the metric (indeed, only 2D metrics are studied in [51]). In particular, we are looking for a transformation allowing to carry

$$ds_{(2)}^2 = -\frac{\gamma^2}{n^2} \left[ -(c^2 - n^2 v^2) dt^2 - 2v(1 - n^2) dt dx + \left( n^2 - \frac{v^2}{c^2} \right) dx^2 \right] \quad (\text{A1})$$

into the form

$$ds_{\text{acoustic}}^2 = -\Omega^2 [-(\tilde{c}^2 - \tilde{v}^2) dt^2 - 2\tilde{v} dt d\tilde{x} + d\tilde{x}^2], \quad (\text{A2})$$

where  $\tilde{v}$  plays the role of fluid velocity (in general depending on  $t, \tilde{x}$ ) and  $\tilde{c}$  is the local speed of sound, assumed to be constant as in [51]. This remapping of the optical metric into an acoustic one is implemented by the identifications

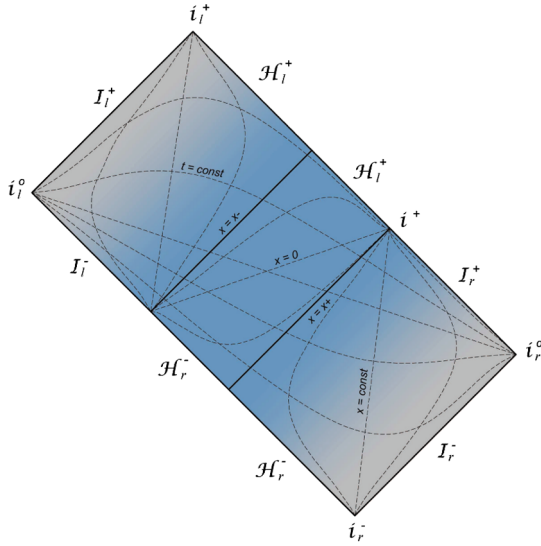


FIG. 5 (color online). Penrose diagram for the analogue metric (2).

$$\tilde{c} = c, \quad (\text{A3})$$

$$\tilde{v} = \gamma^2 v \frac{n^2 - 1}{n}, \quad (\text{A4})$$

$$\Omega^2 = \frac{1}{\gamma^2} \frac{1}{n^2 - \frac{v^2}{c^2}}, \quad (\text{A5})$$

$$\frac{d\tilde{x}}{dx} = -\gamma^2 \frac{n^2 - \frac{v^2}{c^2}}{n}. \quad (\text{A6})$$

It is straightforward to check that the horizon condition  $\tilde{v} = c$  is equivalent to the condition  $n = \frac{c}{v}$ , i.e. the solution of (9). By defining

$$d\tilde{u} = dt - \frac{d\tilde{x}}{\tilde{c} + \tilde{v}}, \quad (\text{A7})$$

$$d\tilde{w} = dt + \frac{d\tilde{x}}{\tilde{c} - \tilde{v}}, \quad (\text{A8})$$

as in [51], one can also easily draw the Penrose diagram of our spacetime (see Fig. 5), which is analogous to that of [51] Fig. 28. It is also straightforward to verify that the surface gravity can be calculated also by means of the formula

$$\kappa_+ = \tilde{c} \left| \frac{\partial}{\partial \tilde{x}} (\tilde{c} - \tilde{v}) \right|_{x=x_+} \quad (\text{A9})$$

(cf. [32]).

## APPENDIX B: THE HAWKING TEMPERATURE AND ITS CONFORMAL INVARIANCE

We consider the Euclidean version of the metric and find that near the horizon the  $ct - x$  part of the metric behaves like a cone which becomes a flat plane only if a special

choice of an angle parameter, to be related to the inverse of the Hawking temperature, is chosen.

The following rescaling to dimensionless variables is performed:

$$c\tau \in (0, \beta) \mapsto \psi = \frac{2\pi}{\beta} c\tau \in (0, 2\pi) \quad (\text{B1})$$

$$x \in (x_+, \infty) \mapsto \bar{x} = \frac{2\pi}{\beta} x \in \left( \frac{2\pi}{\beta} x_+, \infty \right), \quad (\text{B2})$$

where  $\tau$  stays for the Euclidean time. Moreover, the following (local) diffeomorphism on the  $ct - x$  part of the metric is introduced:

$$\begin{aligned} ds_{(2)}^2 &= -\left(\frac{\beta}{2\pi}\right)^2 \left[ \frac{1}{n^2(\bar{x})} g_{\tau\tau}(\bar{x}) d(c^2\tau^2) + \frac{1}{g_{\tau\tau}(\bar{x})} d\bar{x}^2 \right] \\ &= -\left(\frac{\beta}{2\pi}\right)^2 H^2(y) (y^2 d\psi^2 + dy^2), \end{aligned} \quad (\text{B3})$$

whence

$$\frac{1}{n^2(\bar{x})} g_{\tau\tau}(\bar{x}) = y^2 H^2(y), \quad (\text{B4})$$

$$\frac{1}{g_{\tau\tau}(\bar{x})} \left( \frac{d\bar{x}}{dy} \right)^2 = H^2(y). \quad (\text{B5})$$

The above diffeomorphism is such that the  $ct - x$  part of the metric near the horizon can, at least locally, be made conformal to a 2D plane with a suitable choice of  $\beta$ , with conformal factor  $(\frac{\beta}{2\pi})^2 H^2(y)$ . As a consequence, one finds (taking the positive sign)

$$\frac{2\pi}{\beta} \frac{dx}{dy} = \frac{1}{y} \frac{1}{n(x)} g_{\tau\tau}(x), \quad (\text{B6})$$

i.e.

$$y = A \exp\left(\frac{2\pi}{\beta} \int^x dz n(z) \frac{1}{g_{\tau\tau}(z)}\right), \quad (\text{B7})$$

where  $A$  is an integration constant. The diffeomorphism is defined to be regular if we can include in the manifold also the point  $y = 0$ , and then we have to require

$$\lim_{y \rightarrow 0^+} H^2(y) = \lim_{y \rightarrow 0^+} \frac{1}{y^2} g_{\tau\tau}(x(y)) = h_0^2, \quad (\text{B8})$$

where  $h_0^2$  is a finite positive constant. The above limit is equivalent to

$$\lim_{x \rightarrow x_+} \frac{1}{y^2(x)} g_{\tau\tau}(x). \quad (\text{B9})$$

Near the horizon one finds

$$y^2 \sim (x - x_+)^{(2\pi/\beta)(c^2/v^2)(1/\gamma^2)(1/(-\frac{dn}{dx}(x_+))} \quad (\text{B10})$$

and, as a consequence, one has to choose

$$\beta = 2\pi \frac{c^2}{v^2} \frac{1}{\gamma^2 - \frac{dn}{dx}(x_+)} =: \hbar c \beta_h. \quad (\text{B11})$$

The above method can be used also to confirm that the temperature does not depend on the (static) conformal factor. Indeed, an overall conformal factor  $\Omega^2(x)$  which is finite and nonvanishing at the horizons ( $\lim_{x \rightarrow x_{\pm}} \Omega^2(x) = b_{\pm}^2 > 0$ ) does not modify Eq. (B10), as may be realized using Eqs. (B8) and (B9).

### APPENDIX C: EXACT ANALYSIS OF EQUATION (26) NEAR THE SINGULARITY

We confine ourselves to discuss the root  $u_+$ , since all considerations can be trivially extended to  $u_-$ . Then, we rewrite Eq. (26) as

$$A''(u) + \frac{P(u)}{u - u_+} A'(u) + \frac{Q(u)}{u - u_+} A(u) = 0, \quad (\text{C1})$$

where we have introduced the functions  $P(u)$  and  $Q(u)$  (holomorphic in the disk  $|u - u_+| < |u_+ - u_-|$ ):

$$P(u) = 2ik_w \frac{c^2 + n^2(u)v^2}{c^2 - n^2(u)v^2}, \quad (\text{C2})$$

$$Q(u) = -\left(k_w^2 + \frac{k_y^2 + k_z^2}{1 - n^2(u)\frac{v^2}{c^2}}\right). \quad (\text{C3})$$

In a neighborhood of  $u_+$  Eq. (26) writes, to leading order in  $u - u_+$ , as

$$A''(u) - \frac{2ik_w c}{vn'(u_+)} \frac{1}{u - u_+} A'(u) + \frac{(k_y^2 + k_z^2)c}{2vn'(u_+)(u - u_+)} A(u) = 0, \quad (\text{C4})$$

where the prime denotes derivation with respect to the lab variable  $u$ . The indicial equation is  $\alpha(\alpha - 1) - \alpha \frac{2ik_w c}{vn'(u_+)} = 0$ , with roots  $\alpha_1 = 0$  and  $\alpha_2 = 1 + \frac{2ik_w c}{vn'(u_+)}$ . Therefore, in the neighborhood of  $u_+$ , Eq. (26) has two linearly independent solutions of the form

$$A^{(i)}(u) = (u - u_+)^{\alpha_i} \sum_{n=0}^{\infty} c_n^{(i)} (u - u_+)^n, \quad (\text{C5})$$

$i = 1, 2$ , where the series define holomorphic functions in the disc  $|u - u_+| < |u_+ - u_-|$  and whose coefficients can be obtained recursively from the equation  $c_n^{(i)}(\alpha_i + n)(\alpha_i + n - 1 + p_0) + \sum_{r=0}^{n-1} ((\alpha_i + r)p_{n-r} + q_{n-r-1})c_r^{(i)} = 0$ , where the  $p_k, u_k$  ( $k = 0, 1, 2, \dots$ ) are the coefficients of the expansion about  $u_+$  of  $P(u)$  and  $Q(u)$ , respectively. Of particular interest is the solution (25) corresponding to  $\alpha = \alpha_2$ , which for  $u > u_+$  has the form

$$F_{\alpha_2}(u, w, y, z) = \xi(u) e^{i(2k_w c/vn'(u_+)) \log(u - u_+) + ik_w w + ik_y y + ik_z z}, \quad (\text{C6})$$

where  $\xi(u)$  is holomorphic in the neighborhood of  $u = u_+$  and vanishes as  $u \rightarrow u_+$ : indeed, we have reabsorbed in it the factor  $(u - u_+)$  associated with the real term in  $\alpha_2$ , in such a way that

$$\xi(u) = (u - u_+) \eta(u), \quad (\text{C7})$$

where  $\eta(u)$  is holomorphic in the neighborhood of  $u = u_+$  and  $\eta(u) = c_0 + c_1(u - u_+) + \dots$  as  $u \rightarrow u_+$ . We stress the presence of the logarithmic divergence of the phase as  $u$  approaches  $u_+$  even in this approach. Also, by comparison with the study of the solutions contained in Sec. III A, we can infer that the above solution corresponds to the (exact) expansion of  $F_{\mathbf{k}}^+(u, w, y, z)$  near  $u = u_+$ .

### APPENDIX D: ANALYTIC CONTINUATIONS

We are interested in the Bogoliubov coefficient  $\alpha_{\mathbf{k}\mathbf{k}'}$ , which relates the positive frequencies between the initial *in* state:

$$F_{\mathbf{k}}^{\text{in}} = e^{ik_x x + ik_y y + ik_z z - i\omega t}, \quad (\text{D1})$$

and final *out* state  $F_{\mathbf{k}}^+$ :

$$F_{\mathbf{k}}^+ = \theta(u - u_+) \xi_{\mathbf{k}}(u) e^{i(\sigma \log(u - u_+) + k_w w + k_y y + k_z z)}, \quad (\text{D2})$$

where we have introduced the shorthand notation  $\sigma = \frac{2c}{n'(u_+)v} k_w$  and, in agreement with the results obtained in Appendix C, we have introduced also the analytic part  $\xi(u)$ . It will be evident that, at least in the large frequency limit, this  $\xi(u)$  cannot affect the thermal character of particle emission.

Let us start with the computation of  $\alpha_{\mathbf{k}\mathbf{k}'}$ , which, apart from a factor which will not affect our goal, which consists in the deduction of (D7), is given by

$$\alpha_{\mathbf{k}\mathbf{k}'} \propto \delta^2(\mathbf{k}_{\perp} - \mathbf{k}'_{\perp}) \int_0^{\infty} \xi(u) u^{i\sigma} e^{-ik'_u u} du, \quad (\text{D3})$$

where we have shifted the variable  $u$  so that  $u_+$  is mapped on 0. Since  $2k'_u = (k'_x + \omega'/v) > 0$ , we see that the second exponential factor is rapidly decreasing at infinity when  $\text{Im}(u) < 0$ . Consider the path  $\Gamma$  starting from 0 to  $R > 0$  along the real line, then following the arc of radius  $R$  clockwise until  $-iR$ , and finally coming back from  $-iR$  to 0 along the imaginary axis. As the integrand is analytic inside the region bounded by the path, the integral along  $\Gamma$  vanishes for any positive value of  $R$ . Then, taking the limit  $R \rightarrow +\infty$  we then see that the integral along the positive real axis is equal to the integral along the negative imaginary axis and

$$\alpha_{\mathbf{k}\mathbf{k}'} \propto e^{(\pi/2)\sigma} \frac{\delta^2(\mathbf{k}_{\perp} - \mathbf{k}'_{\perp})}{i(k'_u)^{1+i\sigma}} \int_0^{\infty} dt \xi\left(\frac{-it}{k'_u}\right) t^{i\sigma} e^{-t}. \quad (\text{D4})$$

We can consider the limit as  $k'_u \gg 1$  and approximate  $\xi$  for small values of its argument. Then we obtain

$$\alpha_{\mathbf{k}\mathbf{k}'} \propto e^{(\pi/2)\sigma} \frac{\delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp)}{(k'_u)^{2+i\sigma}} \Gamma(2 + i\sigma), \quad (\text{D5})$$

where  $\Gamma$  is the Euler Gamma function.

On the other hand, in order to calculate  $\beta_{\mathbf{k}\mathbf{k}'}$ , it is sufficient to revert the sign of  $k'_u$  and  $\mathbf{k}'$  in Eq. (D3). This time, the integral can be calculated along the positive imaginary axis, since now the term relative to the integra-

tion on the arc vanishes for  $\text{Im}(u) > 0$ . Thus, rotating the path counterclockwise, we obtain

$$\beta_{\mathbf{k}\mathbf{k}'} \propto e^{-(\pi/2)\sigma} \frac{\delta^2(\mathbf{k}_\perp + \mathbf{k}'_\perp)}{(k'_u)^{2+i\sigma}} \Gamma(2 + i\sigma). \quad (\text{D6})$$

By comparing Eqs. (D5) and (D6), it is easy to verify that

$$\sum_{\mathbf{k}'} |\alpha_{\mathbf{k}\mathbf{k}'}|^2 = e^{2\pi\sigma} \sum_{\mathbf{k}'} |\beta_{\mathbf{k}\mathbf{k}'}|^2, \quad (\text{D7})$$

which is Eq. (70).

- 
- [1] S. W. Hawking, *Nature (London)* **248**, 30 (1974).
  - [2] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
  - [3] C. Kiefer, in *Classical and Quantum Black Holes*, edited by P. Fré, V. Gorini, G. Magli, and U. Moschella (IOP Publishing, Bristol, 1999).
  - [4] M. Visser, *Phys. Rev. Lett.* **80**, 3436 (1998).
  - [5] M. Visser, *Int. J. Mod. Phys. D* **12**, 649 (2003).
  - [6] C. Barcelo, S. Liberati, S. Sonogo, and M. Visser, *Phys. Rev. Lett.* **97**, 171301 (2006).
  - [7] W. G. Unruh, *Phys. Rev. Lett.* **46**, 1351 (1981).
  - [8] C. Barcelò, S. Liberati, and M. Visser, *Living Rev. Relativity* **8**, 12 (2005).
  - [9] J. Macher and R. Parentani, *Phys. Rev. A* **80**, 043601 (2009).
  - [10] W. Gordon, *Ann. Phys. (Leipzig)* **377**, 421 (1923).
  - [11] U. Leonhardt and P. Piwnicki, *Phys. Rev. A* **60**, 4301 (1999).
  - [12] U. Leonhardt and P. Piwnicki, *Phys. Rev. Lett.* **84**, 822 (2000).
  - [13] R. Schutzhold, G. Plunien, and G. Soff, *Phys. Rev. Lett.* **88**, 061101 (2002).
  - [14] I. H. Brevik and G. Halmes, *Phys. Rev. D* **65**, 024005 (2001).
  - [15] V. A. De Lorenci and R. Klippert, *Phys. Rev. D* **65**, 064027 (2002).
  - [16] V. A. De Lorenci and M. A. Souza, *Phys. Lett. B* **512**, 417 (2001).
  - [17] M. Novello and J. M. Salim, *Phys. Rev. D* **63**, 083511 (2001).
  - [18] M. Marklund, D. Anderson, F. Cattani, M. Lisak, and L. Lundgren, *Am. J. Phys.* **70**, 680 (2002).
  - [19] V. A. De Lorenci, R. Klippert, and Yu. N. Obukhov, *Phys. Rev. D* **68**, 061502 (2003).
  - [20] M. Novello, S. E. Perez Bergliaffa, J. Salim, V. De Lorenci, and R. Klippert, *Classical Quantum Gravity* **20**, 859 (2003).
  - [21] M. Novello and S. E. Perez Bergliaffa, in *COSMOLOGY AND GRAVITATION: Xth Brazilian School of Cosmology and Gravitation; 25th Anniversary (1977-2002)*, edited by Mário Novello, and Santiago E. Perez Bergliaffa, AIP Conf. Proc. No. 668 (AIP, New York, 2003).
  - [22] W. G. Unruh and R. Schutzhold, *Phys. Rev. D* **68**, 024008 (2003).
  - [23] R. Schutzhold and W. G. Unruh, *Phys. Rev. Lett.* **95**, 031301 (2005).
  - [24] T. G. Philbin, C. Kuklewicz, S. Robertson, S. Hill, F. König, and U. Leonhardt, *Science* **319**, 1367 (2008).
  - [25] D. Faccio, S. Cacciatori, V. Gorini, V. G. Sala, A. Averchi, A. Lotti, M. Kolesik, and J. V. Moloney, *Europhys. Lett.* **89**, 34004 (2010).
  - [26] F. Belgiorno, S. L. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V. G. Sala, and D. Faccio, *Phys. Rev. Lett.* **105**, 203901 (2010).
  - [27] J. Schwinger, *Proc. Natl. Acad. Sci. U.S.A.* **89**, 4091 (1992); **89**, 11118 (1992); **90**, 958 (1993); **90**, 2105 (1993); **90**, 4505 (1993); **90**, 7285 (1993); **91**, 6473 (1994).
  - [28] D. Faccio, F. Belgiorno, S. Cacciatori, M. Clerici, V. Gorini, G. Ortenzi, L. Rizzi, E. Rubino, V. G. Sala, in *Nonlinear Optics and Applications IV*, edited by Benjamin J. Eggleton, Alexander Luis Gaeta, and Neil G. R. Broderick (Proc. of SPIE Vol. 7728, 77280M (2010)).
  - [29] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
  - [30] The factor  $c^4$  appears since we are using standard units in place of the customary natural ones.
  - [31] T. Jacobson and G. Kang, *Classical Quantum Gravity* **10**, L201 (1993).
  - [32] M. Visser, *Classical Quantum Gravity* **15**, 1767 (1998).
  - [33] A. Coutant and R. Parentani, *Phys. Rev. D* **81**, 084042 (2010).
  - [34] W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
  - [35] S. Flugge, *Practical Quantum Mechanics* (Springer, New York, 1999).
  - [36] W. G. Unruh, *Phys. Rev. D* **51**, 2827 (1995).
  - [37] S. Corley and T. Jacobson, *Phys. Rev. D* **54**, 1568 (1996).
  - [38] T. Jacobson, *Phys. Rev. D* **53**, 7082 (1996).
  - [39] J. Macher and R. Parentani, *Phys. Rev. D* **79**, 124008 (2009).
  - [40] R. Balbinot, A. Fabbri, S. Fagnocchi, A. Recati, and I. Carusotto, *Phys. Rev. A* **78**, 021603 (2008).
  - [41] S. Corley and T. Jacobson, *Phys. Rev. D* **59**, 124011 (1999).
  - [42] S. Finazzi and R. Parentani, *New J. Phys.* **12**, 095015 (2010).
  - [43] U. Leonhardt and T. G. Philbin, in *Quantum Analogues: From Phase Transitions to Black Holes and Cosmology*

- edited by W.G. Unruh and R. Schützhold (Springer-Verlag, Berlin, 2007).
- [44] A. Recati, N. Pavloff, and I. Carusotto, *Phys. Rev. A* **80**, 043603 (2009).
- [45] B. Huttner and S.M. Barnett, *Phys. Rev. A* **46**, 4306 (1992); B. Huttner, J.J. Baumberg, and S.M. Barnett, *Europhys. Lett.* **16**, 177 (1991).
- [46] S.L. Cacciatori, F. Belgiorno, V. Gorini, G. Ortenzi, L. Rizzi, V.G. Sala, and D. Faccio, *New J. Phys.* **12**, 095021 (2010).
- [47] W.G. Unruh and R. Schutzhold, *Phys. Rev. D* **68**, 024008 (2003).
- [48] L.G. Sutorp and A.J. van Wonderen, *Europhys. Lett.* **67**, 766 (2004).
- [49] S. Robertson and U. Leonhardt, *Phys. Rev. A* **81**, 063835 (2010).
- [50] P.W. Milonni, *J. Mod. Opt.* **42**, 1991 (1995).
- [51] C. Barcelo, S. Liberati, S. Sonego, and M. Visser, *New J. Phys.* **6**, 186 (2004).